LA-UR-03-6999

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Title: VERIFICATION OF AN ASCI SHAVANO PROJECT HYDRODYNAMICS ALGORITHM

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Date: September 2003

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September 30, 2003

LA-UR-03-6999

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Abstract

In this report we describe a method for verification analysis of a 2-D Lagrangian, compressible hydrodynamics algorithm based on an unstructured mesh. The metrics for this verification study are the parameters associated with asymptotic convergence analysis; in particular, the asymptotic convergence rate is highlighted as the main gauge of verification. For the problems under consideration, the convergence analysis is complicated by two factors: (1) the computational grid evolves in time as part of the Lagrangian formulation of the underlying equations, and (2) the cell-based, characteristic length scales of the spatial discretization vary throughout the unstructured mesh. We present the necessary background on the relevant mathematical and numerical issues to motiviate this analysis and provide examples of this process on two problems: one with a smooth solution and the other with a discontinuous solution. The results demonstrate the viability of this verification analysis approach for unstructured mesh calculations.

1 Introduction

In this report we describe the process of asymptotic convergence analysis for Lagrangian, compressible hydrodynamics algorithms that involve unstructured, dynamically evolving meshes. This analysis provides numerical estimates of the asymptotic convergence rate, which provides a repeatable, quantified metric for software implementations of certain numerical algorithms. Such quantified code-convergence characteristics form a foundational part of the evidence by which the quality of such simulation software is established and tracked.

Convergence analysis of numerical solutions to partial differential equations (PDEs) is typically performed on algorithms for which the equations have been discretized on a uniform, fixed mesh [15, 17, 18, 9]. Such regular meshes greatly simplify both the method and implementation of the asymptotic convergence analysis. These meshes are associated, e.g., with Eulerian hydrodynamics algorithms for the flow of compressible fluids.

Lagrangian, staggered-grid hydrodynamics algorithms differ significantly from Eulerian algorithms. Lagrangian methods have a rich history [19] and are known for their ability to model systems with one-dimensional, convergent fluid-flow and strong shocks. These algorithms have been adapted to work in two and three dimensions and on unstructured meshes [2, 4, 5].

It is outside of the scope of this report to provide a detailed explanation of these multi-dimensional algorithms or the gas dynamics equations that they model; see the above references for details. There are, however, several key features of these algorithms that impact verification analysis. The first is an unstructured mesh: the mesh is defined by a collection of control nodes that are topologically organized into cells. The mesh is unstructured in the sense that individual cells may be constructed from an arbitrary, non-uniform number of nodes. This organization is, however, generally constant in time. The second important aspect is staggered variable placement: the velocity field is centered at the control nodes, while the thermodynamic state variables (viz., density, pressure, and internal energy) are centered at cells and considered constant and uniform within the cell. The last key feature is the Lagrangian frame of reference: the gas dynamics equations are framed such that the control nodes' positions evolve with time and are thought of as being imbedded within the fluid.

Although these mesh and reference-frame characteristics allow for straightforward discretization of the gas dynamaics equations, they present challenges in the verification of the underlying algorithms. The unstructured mesh and Lagrangian reference frame imply that cells can be arbitrary polyhedra with time-varying dimensions, which renders the definition of the characterisic resolution scale of the computational mesh ambiguous. The staggered grid placement of variables means that node- and cell-centered variables require different methods by which to evaluate the integrals representing the norms of errors in the calculated quantities.

In this report, we describe how to overcome these fundamental challenges by adapting the method of aysmptotic convergence analysis from its well established form (i.e., a fixed, structured mesh wighin an Eulerian reference frame) to application on Lagrangian hydrodynamics algorithms.[1] The fundamental notions and procedures are presented in §2, which contains an explanation of global asymptotic convergence analysis for Lagrangian, unstructured-mesh algorithms. §3 contains convergence analysis for a 2-D smooth problem, while §4 presents the corresponding results for a 2-D nonsmooth problem. The computational results in these two sections were obtained with ASCI Shavano project codes. We summarize the contents of this report in §5.

2 Global Convergence Analysis on Unstructured Meshes

The axiomatic premise of asymptotic convergence analysis is that the norm of the difference between the exact and computed solutions can be expanded as a function of some measure of the spatial and temporal zone sizes. In the following subsections, we describe this assumption in detail and provide algorithmic descriptions of how these concepts are implemented numerically for Lagrangian unstructured-mesh algorithms.

2.1 Unstructured-Mesh Convergence in One Dimension

For PDEs discretized in time and a single spatial coordinate, the fundamental ansatz of asymptotic convergence analysis assumes the following form (see, e.g., [8]):

$$\begin{aligned} ||\xi^* - \xi|| = \mathcal{A}\ (\Delta\tilde{x})^q + \mathcal{B}\ \left(\Delta\tilde{t}\right)^r \\ + o\Big((\Delta\tilde{x})^q\,, \left(\Delta\tilde{t}\right)^r\Big)\,, \end{aligned} \tag{1}$$

where ξ^* is the *exact* solution value; ξ is the value computed on the grid for which $\Delta\tilde{x}$ and $\Delta\tilde{t}$ provide measures, respectively, of the spatial zone size and temporal step size of the computational mesh; $\mathcal{A}$ is the *spatial convergence coefficient*; q is the *spatial convergence rate*; $\mathcal{B}$ is the *temporal convergence coefficient*; and r is the *temporal convergence rate*. By the notation "$o\Big((\Delta\tilde{x})^q\,, \left(\Delta\tilde{t}\right)^r\Big)$" we mean terms that approach zero faster than $(\Delta\tilde{x})^q$ and $\left(\Delta\tilde{t}\right)^r$ as both $\Delta\tilde{x}$ and $\Delta\tilde{t}$ becomes vanishingly small (i.e., as $\Delta\tilde{x}$, $\Delta\tilde{t} \to 0^+$). In the following, we also refer to these vanishingly small terms as "higher order terms" or H.O.T. We note parenthetically that one could employ the Method of Manufactured Solutions (MMS) to obtain exact solutions of an *inhomogenous*

[1] The approach described herein would also be applicable, with minor modification, to ALE (Arbitrary Lagrangian-Eulerian) [11] and AMR (Adaptive, Mesh Refinement) algorithms.

set of equations that is related to the equations of interest; Salari & Knupp [18] and Knupp & Salari [9] provide introductions to this technique.

The "measures" of the spatial and temporal computational meshes in Eq. 1, i.e., $\Delta\tilde{x}$ and $\Delta\tilde{t}$, respectively, have a particular and important meaning. For example, in a dynamically evolving 1-D mesh, the computational domain will consist of cells of various lengths; these cells likely become elongated or shortened in the course of the calculation. Verification analysis, however, requires a single, scalar characterization of the 1-D extent of all mesh cells. For example, three measures of the spatial extent of the zones, are the minimum, maximum, and mean over the set of the lengths of all zones in the mesh. The identification of characteristic scalar spatial and temporal scales for a calculation involving irregular cell sizes and nonuniform timesteps is one significant feature that distinguishes the present verification analysis.

The norm in Eq. 1 is the functional $||\cdot||$ that maps its argument, a function, to the non-negative real numbers and obeys the appropriate functional analytic properties. The standard definition for the L_p functional norm of a function f that depends on a single spatial variable on the interval $[a,b]$ is

$$||f||_p \equiv \left[\frac{1}{L}\int_a^b dx\,|f(x)|^p\right]^{1/p} , \tag{2}$$

where $L \equiv b-a$; according to this definition, the units of $||f||_p$ are identical to the units of f. From Eq. 2, the standard L_1, L_2, and L_∞ norms are defined as

$$||f||_1 \equiv \frac{1}{L}\int_a^b dx\,|f(x)|\,, \tag{3}$$

$$||f||_2 \equiv \sqrt{\frac{1}{L}\int_a^b dx\,|f(x)|^2}\ , \quad \text{and} \tag{4}$$

$$||f||_\infty \equiv \max_{x\in[a,b]}|f(x)|\,, \tag{5}$$

the last of which is independent of any integral and is trivial to evaluate numerically.

In the application at hand, the argument of the norm is not an abstract function, but a collection of numerical values on a computational mesh. To numerically approximate the integral on the interval $[a,b]$, first assume there are N_c points x_i, $i = 1,\ldots,N_c$, one each at the geometric center[2] of a zone of length Δx_i. This assumption applies, e.g., in the case of the density and pressure fields for typical Lagrangian, staggered-grid hydrodynamics codes. The

[2]More generally, the position of this single quadrature point need not be at the cell center but could be an appropriately defined cell centroid.

simplest quadrature scheme is then to approximate such integrals as follows:

$$\frac{1}{L}\int_a^b dx\, g(x) \doteq \frac{1}{b-a}\sum_{i=1}^{N_c} g(x_i)\,\Delta x_i\,. \tag{6}$$

With this assumption, the L_1 and L_2 in Eqs. 3 and 4 are evaluated as:

$$||f||_1 \equiv \frac{1}{L}\sum_{i=1}^{N_c} |f(x_i)|\,\Delta x_i\,, \tag{7}$$

$$||f||_2 \equiv \sqrt{\frac{1}{L}\sum_{i=1}^{N_c} |f(x_i)|^2\,\Delta x_i}\;. \tag{8}$$

It is quantities such as these that are used to generate numerical estimates of the left-hand side of Eq. 1.

Staggered-grid Lagrangian hydrodynamics algorithms [19] define the velocity field at the N_v vertices of the computational cells, as opposed to the N_c cell centers. Consequently, a slightly different procedure is required to numerically approximate the integrals of these values. Assume now that there are N_v points, x_i, $i = 1,\ldots,N_v$, one each at the mesh $\underline{v}$ertex, which lies between the two cells whose extents are $\Delta x_{i-1/2}$ and $\Delta x_{i+1/2}$. The approach we follow approximates the normalized integral of such function values as:

$$\frac{1}{L}\int_a^b dx\, g(x) \doteq \frac{1}{\mathcal{L}}\sum_{i=1}^{N_v} g(x_i)\left(\Delta x_{i-1/2} + \Delta x_{i+1/2}\right)\,, \quad \text{where} \tag{9}$$

$$\mathcal{L} \equiv \sum_{i=1}^{N_v}\left(\Delta x_{i-1/2} + \Delta x_{i+1/2}\right)\,, \tag{10}$$

and $\Delta x_{i-1/2}$ ($\Delta x_{i+1/2}$) is the length of the zone to the left (right) of x_i that has x_i as a vertex. Equations 9 and 10 assume an effective vertex-centered cell-size consisting of the sum of both cells adjacent to the vertex. This over-assignment of the cell sizes that weight $g(x_i)$ in Eq. 9, however, is normalized by the corresponding effective total length as computed in Eq. 10. While other options for defining vertex-centered cell sizes in these approaches are possible (using, e.g., vertex control volumes), the approach expressed in Eqs. 9 and 10 suffices for the present demonstration of Lagrangian unstructured-mesh convergence analysis.

According to the fundamental assumption in Eq. 1, the hallmark of a convergent solution is a positive convergence rate: for positive values of q and r in the above relation, a finer mesh (either spatial or temporal) implies that the difference between the computed and exact solutions is smaller (i.e., the norm on the left-hand side of the above equation decreases). Expressed another way, the computed solution converges to the exact solution if a smaller error (as measured by the norm) obtains as more "resources" (e.g., a more refined grid) are applied in the numerical solution.

A numerical solution computed with a higher convergence rate algorithm on a given grid may result in a more faithful approximation to the correct solution than a numerical solution computed with a lower convergence rate algorithm on the same (or comparable) grid. *All else being equal*, numerical algorithms with higher convergence rates are more desirable than lower convergence rates. Typically, however, "all else" is *not* equal, as higher convergence rate algorithms typically consume greater computational resources than algorithms with lower convergence rates; this tradeoff between accuracy and efficiency is examined in detail by Rider et al. [16].

2.2 Unstructured-Mesh Convergence in Higher Dimensions

For higher dimensions, the above convergence ansatz generalizes naturally in the case of a uniform, static mesh, such as those employed, e.g., in the Eulerian formulation of the gas dynamic equations; see, e.g., [8]. In the present work, however, we do not have the convenience of that uniformity. Consequently, we generalize the error ansatz of Eq. 1, viz.,

$$||\xi^* - \xi|| = \mathcal{A}\ (\Delta\tilde{x})^q + \mathcal{B}\ \left(\Delta\tilde{t}\right)^r + o\Big((\Delta\tilde{x})^q\,, \left(\Delta\tilde{t}\right)^r\Big)\,, \tag{11}$$

to higher dimensions by requiring that the symbol $\tilde{x}$ represent a measure of the generalized 1-D spatial extent for the higher-dimensional mesh cells; the other parameters are as described in the previous section. The issue of assigning characteristic length and time scales for a mesh consisting of irregular cells and non-uniform timesteps lies at the heart of convergence analysis criteria for unstructured, dynamically evolving meshes.

Some explanation of the important concepts related to the unstructured mesh and embodied in this description is in order. By "1-D spatial extent" we mean that this measure has units identical to a single spatial dimension; for example, this measure must have units, e.g., of cm and not cm^2 in 2-D or cm^3 in 3-D computational domains. Correspondingly, by "generalized" 1-D spatial extent we mean that a 1-D measure is constructed from the n-D data; for example, in 2-D, where the zones may be, e.g., generalized quadrilaterals, a 1-D measure for a given zone might be the square root of the area of that zone. For an entire mesh, there will be a set of such values, one corresponding to each mesh cell. Lastly, we seek to motivate the notion behind the term "measure" of the generalized 1-D spatial extent. In a compressible Lagrangian calculation, the initial mesh likely becomes distorted as the calcuation proceeds, with mesh cells potentially becoming highly non-uniform geometrically. Three measures of the "1-D-ness" of the zones, applicable over the entire mesh, would be the square roots of the minimum, maximum, and mean over the set of areas of all zones in the mesh; one can easily arrive at other such measures. In a fixed, uniform mesh calculation, such distinctions are moot, as these measures all reduce to the same value for the entire mesh for the entire simulation.

The descriptions provided above for approximating 1-D norms extend naturally to the higher dimensions. Specifically, the definition for the L_p functional norm of the function f defined on the set $\Omega \subset \mathbb{R}^n$ is

$$||f||_p \equiv \left[\frac{1}{\mu(\Omega)} \int_\Omega d^n\mathbf{x}\, |f(\mathbf{x})|^p\right]^{1/p} , \tag{12}$$

where the notation $d^n\mathbf{x}$ indicates that the argument $\mathbf{x}$ is an n-vector (i.e., $\mathbf{x} \in \mathbb{R}^n$), and $\mu(\Omega)$ is the measure of the set Ω (e.g., the area in 2-D or the volume in 3-D). Based on this definition, the L_1, L_2, and L_∞ norms are defined as

$$||f||_1 \equiv \frac{1}{\mu(\Omega)} \int_\Omega d^n\mathbf{x}\, |f(\mathbf{x})| , \tag{13}$$

$$||f||_2 \equiv \sqrt{\frac{1}{\mu(\Omega)} \int_\Omega d^n\mathbf{x}\, |f(\mathbf{x})|^2} , \quad \text{and} \tag{14}$$

$$||f||_\infty \equiv \max_{\mathbf{x}\in\Omega} |f(\mathbf{x})| , \tag{15}$$

the last of which is, again, independent of any integral.

We illustrate the multidimensional norm by considering the case $n = 2$. In this case, $\Omega \subset \mathbb{R}^2$, a closed region in the plane. Assume that this domain is the union of zones, ω_α, $\alpha = 1, \ldots, N_c$, that comprise the computational mesh, such that (i) $\alpha \neq \beta \Rightarrow \omega_\alpha \cap \omega_\beta = \partial\omega_{\alpha\beta}$, which is either the empty set (if the zones are not contiguous) or two vertices and one curve (typically, a line) between them (if the zones are contiguous); and (ii) $\cup_\alpha \omega_\alpha = \Omega$, i.e., the whole mesh. Let this set of N_c zones be associated with N_v vertices, $V_1, \ldots, V_{N_v}$.

The field of numerical quadrature addresses questions related to the numerical approximation of integrals; given the data we have from the hydrocode, we consider only approximations based locally on a single function evaluation. For cell-centered values, the simplest of these techniques is to approximate the general 2-D integral as follows:

$$\frac{1}{\mu(\Omega)} \int_\Omega d^2\mathbf{x}\, f(\mathbf{x}) \doteq \frac{1}{\mu(\Omega)} \sum_{\alpha=1}^{N_c} f(x_\alpha, y_\alpha)\, \mu(\omega_\alpha) . \tag{16}$$

Here, $\mu(\omega_\alpha)$ is the area of subregion (i.e., zone) ω_α, and the total area $\mu(\Omega)$

is the sum of the areas of the zones in the mesh:

$$\mu(\Omega) \doteq \sum_{\alpha=1}^{N_c} \mu\left(\omega_\alpha\right) \ . \tag{17}$$

As in the 1-D case, an important aspect of quadrature schemes is, of course, *where* the function evaluation occurs within each sub-region, i.e., where (x_α, y_α) is located within ω_α. We use the location and function values of the cell-centered variables as provided in the hydrocode output. This assumption and those of Eqs. 13 and 14 imply that the L_1 and L_2 norms become:

$$||f||_1 \equiv \frac{1}{\mu(\Omega)} \sum_{\alpha=1}^{N_c} |f(x_\alpha, y_\alpha)| \, \mu\left(\omega_\alpha\right) \ , \tag{18}$$

$$||f||_2 \equiv \sqrt{\frac{1}{\mu(\Omega)} \sum_{\alpha=1}^{N_c} |f(x_\alpha, y_\alpha)|^2 \, \mu\left(\omega_\alpha\right)} \ , \tag{19}$$

where $\mu(\Omega)$ is evaluated according to Eq. 17. All of these quantities can be evaluated directly with the values output by the hydrocode.

For vertex-centered values, the approach described above to approximate 1-D integrals with weighted averages extends to 2-D as follows:

$$\frac{1}{\mu(\Omega)} \int_\Omega d^2\mathbf{x} \, g(\mathbf{x}) \doteq \frac{1}{\tilde{\mu}(\Omega)} \sum_{\alpha=1}^{N_v} g(x_\alpha, y_\alpha) \, \tilde{\mu}\left(\omega_\alpha\right) \ . \tag{20}$$

In these expressions, $\tilde{\mu}\left(\omega_\alpha\right)$ is given by

$$\tilde{\mu}(\Omega) = \sum_{\alpha=1}^{N_v} \tilde{\mu}(\omega_\alpha) \quad \text{with} \quad \tilde{\mu}\left(\omega_\alpha\right) = \mu\left(\{\omega_\beta : \omega_\beta \cap V_\alpha \neq \emptyset\}\right) \ , \tag{21}$$

where V_α is the αth vertex. In words, $\tilde{\mu}(\Omega)$ is the value determined by looping over all vertices and incrementing this sum at the αth vertex by the area of *all* zones ω_β that share the αth vertex. As described in §2.1 for the 1-D case, this approach assumes an effective vertex-centered cell-size consisting of the sum of all cells adjacent to the vertex. This over-assignment of cell areas, however, is normalized in Eq. 20 by the corresponding effective total area as computed in Eq. 21.

The approximations in Eqs. 20 and 21 can be generalized using more accurate (e.g., multi-point) quadrature schemes. Additionally, one could employ a more precise description for defining vertex-centered cell areas (using, e.g., vertex control volumes). Notwithstanding these considerations, the approach expressed in Eqs. 20 and 21 is adequate to demonstrate convergence analysis for Lagrangian grids.

2.3 Estimation of Spatial Convergence Parameters for Unstructured-Mesh Calculations

We herewith restrict our attention to the case of spatial convergence. Furthermore, we implicitly assume that the spatial discretization error dominates the temporal discretization error. In this situation, the ansatz of Eq. 1 reduces to:

$$||\xi^* - \xi|| = \mathcal{A}\ (\Delta\tilde{x})^q + \text{H.O.T.} \tag{22}$$

Recall that $\tilde{x}$ represents a characteristic measure of the generalized 1-D spatial extent for the set of all mesh cells.

This equation contains two unknown parameters, $\mathcal{A}$ and q. We obtain closed-form solutions for these values by using two computed solutions, one obtained with a (relatively) coarse mesh (subscripted c) and the other with a (relatively) fine mesh[3] (subscripted f), together with the exact solution (superscripted $*$):

$$\begin{aligned} ||\xi^* - \xi_c|| &= \mathcal{A}\ (\Delta\tilde{x}_c)^q + \text{H.O.T.}\,, & (23)\\ ||\xi^* - \xi_f|| &= \mathcal{A}\ (\Delta\tilde{x}_f)^q + \text{H.O.T.} & (24)\end{aligned}$$

Taking the logarithm of each of these equations,

$$\begin{aligned} \log||\xi^* - \xi_c|| &= \log\mathcal{A} + q\ \log(\Delta\tilde{x}_c) + \cdots, & (25)\\ \log||\xi^* - \xi_f|| &= \log\mathcal{A} + q\ \log(\Delta\tilde{x}_f) + \cdots, & (26)\end{aligned}$$

and subtracting them leads to the following expression for the asymptotic convergence rate:

$$q \doteq \log\left(||\xi^* - \xi_c|| / ||\xi^* - \xi_f||\right) / \log\left(\Delta\tilde{x}_c / \Delta\tilde{x}_f\right)\ . \tag{27}$$

Substituting this expression back into Eq. 23 provides a numerical estimate of the convergence coefficient $\mathcal{A}$:

$$\mathcal{A} \doteq ||\xi^* - \xi_c|| / \left(\Delta\tilde{x}_c\right)^q\ ; \tag{28}$$

one obtains a similar expression with an equivalent numerical result by substituting the expression for q back into Eq. 24. Since the norm in this expression is perforce non-negative, and the characteristic length measure is positive, this relation implies that $\mathcal{A}$ must be positive.

Evaluation of Eq. 27 requires the following quantities: (i) the numerical solution computed on a coarse grid, (ii) the exact solution evaluated at corresponding coarse-grid points, (iii) the numerical solution computed on a fine

[3]The description of the meshes as "coarse" or "fine" is in reference to the *initial* mesh only: by the intrinsic nature of Lagrangian calculations, the mesh evolves with the computation, so that *a priori* characterizations of the mesh for $t > 0$ based on the mesh at $t = 0$ are, at best, speculative.

grid, (iv) the exact solution evaluated at corresponding fine-grid points, and (v) numerical characterizations of the coarse and fine meshes (e.g., number of cells N_c, cell areas, number of vertices N_v, etc.). With this information, one evaluates the norms according to the approximation procedures for the required numerical quadratures, as outlined in the previous sections.

Two additional comments are appropriate. First, we impose that the timestep in both "coarse" and "fine" calculations be equal and that the number of computational timesteps in these calcuations likewise be the same; these restrictions ensure that the calculations are evaluated at the identical final time. Second, we observe that, according to the discussion provided in §2.2, different methods by which to assign values to the characteristic lengths $\Delta\tilde{x}_c$ and $\Delta\tilde{x}_f$ may lead to different values for the quantifiable convergence metrics, viz., $\mathcal{A}$ and q. In the following sections, we provide examples of this analysis.

3 Convergence Analysis of a Smooth Problem

In this section, we restrict the notions of the previous section to two dimensions and demonstrate the verification analysis of a hydrodynamics calculation on a non-uniform grid, using a Lagrangian formation of the governing equations. For the numerical computations, an ASCI Shavano project code was employed. The hydrodynamics algorithm is a control volume, staggered-grid, compatible, Lagrangian method [2, 4]. The salient mesh and reference-frame features of this hydrodynamics algorithm are described in §1.

3.1 Exact Solution of a Smooth Problem

The initial conditions for this two-dimensional, Cartesian geometry problem consist of sinusoidal distributions of density, pressure, and velocity that are perturbations about constant, uniform values. For sufficiently small initial amplitude, the sinusoidal density, pressure, and velocity distributions oscillate, undisturbed, as a standing acoustic wave; see, e.g., Landau and Lifschitz [10] or Whitham [20] for details. This solution satisfies the linear acoustics equations, which are first-order linearizations of the full compressible hydrodynamics equations. More precisely, it is assumed that there are uniform, constant density ρ_0, pressure p_0, and velocity (u_0, v_0) fields about which there are small perturbations (in a manner to be made precise). Identifying these perturbations with a prime ($'$), we expand these fields as follows:

$$\rho = \rho_0 + \rho' \,, \tag{29}$$

$$p = p_0 + p' \,, \tag{30}$$

$$(u, v) = (u_0, v_0) + (u', v') \,. \tag{31}$$

The specific internal energy, e, is related to the density and pressure through the standard polytropic gas relation

$$p = (\gamma - 1)\, \rho\, e\,. \tag{32}$$

This relation is used to express the quiescent sound speed, c_0, as

$$c_0^2 = \gamma p_0/\rho_0 = \gamma\, (\gamma - 1)\, e_0\,. \tag{33}$$

It is this quantity that permits an ordering of all the terms in Eqs. 29–31: the primed quantities are small in the sense that

$$\rho'/\rho_0 \ll 1\,, \qquad p'/p_0 \ll 1\,, \qquad \text{and} \qquad u'/c_0,\ v'/c_0 \ll 1\,. \tag{34}$$

Following the development found in references [10] and [20], the solution for the first-order perturbations can be expressed in terms of the potential $\varphi(\mathbf{x}, t)$, which satisfies the linear wave equation:

$$\frac{\partial^2 \varphi}{\partial t^2} - c_0^2 \nabla^2 \varphi = 0\,. \tag{35}$$

The velocity perturbation is the gradient of this potential:

$$(u', v') = \nabla \varphi = \left(\frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial y} \right)\,. \tag{36}$$

The pressure and density perturbations are related to temporal derivatives of the potential as:

$$\rho' = -\frac{\rho_0}{c_0^2} \frac{\partial \varphi}{\partial t}\,, \tag{37}$$

$$p' = -\rho_0 \frac{\partial \varphi}{\partial t}\,. \tag{38}$$

We assign the domain of interest to be the square of unit dimension in the plane, i.e., $\{(x, y) : 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1\}$, and prescribe periodic boundary conditions along the edge of this region. With this assumption, Eq. 35 can be solved analytically using separation of variables. The initial conditions are such that the density and pressure perturbations are identically zero; additionally, the velocity perturbation undergoes one full spatial period along each boundary of the domain and has absolute magnitude proportional to ε, which is small in the sense of the relations in Eq. 34. The temporal dependence is constrained to be oscillatory with angular frequency ω.

With these constraints, the specific solution for the potential is given as:

$$\varphi\, (\mathbf{x}, t) = \frac{\varepsilon}{|\mathbf{k}|}\, \sin(\mathbf{k} \cdot \mathbf{x})\, \cos(\omega t)\,. \tag{39}$$

In this expression, the wavevector $\mathbf{k} \equiv k_x \mathbf{x} + k_y \mathbf{y}$ governs the direction and period of the spatial variation in the solution. From Eqs. 36, 37, and 38, it is

2-D Smooth Problem Solution

	Base $(-)_0$	*Perturbed* $(-)'$
u	0	$\varepsilon \frac{k_x}{k} \cos(k_x x + k_y y) \cos(\omega t)$
v	0	$\varepsilon \frac{k_y}{k} \cos(k_x x + k_y y) \cos(\omega t)$
ρ	1	$\frac{\varepsilon \rho_0 \omega}{k c_0^2} \sin(k_x x + k_y y) \sin(\omega t)$
p	3/10	$\frac{\varepsilon \rho_0 \omega}{k} \sin(k_x x + k_y y) \sin(\omega t)$

Table 1: Closed-form expressions for the base ($_0$) and perturbed ($'$) solutions for the x-component of velocity (u), y-component of velocity (v), the density (ρ), and the pressure (p), where $k \equiv |\mathbf{k}| = \left(k_x^2 + k_y^2\right)^{1/2}$.

2-D Smooth Problem Initial Parameters

γ	k_x	k_y	ω	ε
5/3	2π	2π	2π	10^{-4}

Table 2: Parameter values used in the specification of smooth problem considered: the adiabatic exponent γ used in the polytropic equation of state $p = (\gamma - 1)\,\rho e$, the wavevector $\mathbf{k} = (k_x, k_y)$, the angular frequency ω of the time dependence, together with amplitude of the components of the initial velocity perturbation (ε). These quantities completely specify the potential in Eq. 39 and corresponding fields given in Table 1.

a straightforward task to derive the closed-form expressions for the complete solution to this problem, which is given in Table 1. The reader is reminded that the analytic solution, against which the computed solution is compared, is an exact solution to the Euler equations of gas dynamics *only* in the limit of vanishingly small intial perturbation ampliltude.

3.2 Computed Solution of a Smooth Problem

This standing wave problem was considered in the case of two-dimensional Cartesian geometry. As described above, the problem was assigned on the first quadrant of (x, y)-plane. The initial grid is given by cells determined by K lines parallel to the x-axis and L lines parallel to the y-axis. The configurations considered consisted of $K = L = 10$, 20, and 40, corresponding to 100, 400, and 1600 cells with 121, 441, and 1681 vertices, respectively. The period of the temporal oscillation is unity, so that the initial state is repeated at $t = 0, 1, 2, \ldots$ This problem was run to a final time of 0.2 with uniform, constant timesteps.

Table 3 contains information about the meshes at the final computational

2-D Smooth Problem Mesh Statistics at $t = 0.2$

	Mesh 1	Mesh 2	Mesh 3
N_c	100	400	1600
N_v	121	441	1681
Area/N_c	1.00×10^{-2}	2.50×10^{-3}	6.25×10^{-4}
Mean	1.00×10^{-2}	2.50×10^{-3}	6.25×10^{-4}
Minimum	1.00×10^{-2}	2.50×10^{-3}	6.25×10^{-4}
Maximum	1.00×10^{-2}	2.50×10^{-3}	6.25×10^{-4}
Median	1.00×10^{-2}	2.50×10^{-3}	6.25×10^{-4}

Table 3: Statistics for the smooth problem meshes at $t = 0.2$ for the three resolutions considered: the number of cells, the number of vertices, the mesh area divided by the number of cells, and the mean, minimum, maximum, and median of the areas of all cells in the mesh. All values for a given mesh are identical to the three significant figures quoted; any differences among the various values appeared in the fourth or higher significant figure.

time. This table shows the values of the characteristics corresponding to the total mesh area divided by the number of cells (Area/N_c), and the mean, minimum, maximum, and median over the set of all cell areas in the (x, y)-computational plane. For a given mesh, these various values are identical to the three significant figures tabulated. That all these values are virtually identical illustrates the minimal mesh distortion incurred in the course of this calculation; this behavior is consistent both with what one might expect for standing wave behavior and with the small absolute value of the imposed perturbation.

3.3 Convergence Analysis of a Smooth Problem

An ASCI Shavano project code was used to compute numerical solutions to the compressible gas dynamics equations for the problem described above. We consider asymptotic convergence analysis for two cell-centered quantities, the density and pressure, together with the vertex-centered velocity field. In this section, we describe the outcome of this investigation and catalogue the results in several tables.

The calculated norms of the difference between calculated and exact solutions over the computational meshes (as specified in Table 3) are given in Table 4. Since the problem we consider here is formulated in 2-D Cartesian geometry, the norm is obtained as an estimate of an (x, y)-plane quadrature, the evaluation of which follows in a straightforward manner from the procedure outlined in §2.

These values are used to calculate the corresponding convergence rates and

2-D Smooth Problem Error Norms

		ρ	p	$\lvert(u,v)\rvert$
	L_1	2.5×10^{-6}	1.2×10^{-6}	4.9×10^{-6}
10×10	L_2	2.8×10^{-6}	1.4×10^{-6}	5.4×10^{-6}
	L_∞	3.8×10^{-6}	1.9×10^{-6}	7.6×10^{-6}
	L_1	5.9×10^{-7}	2.9×10^{-7}	1.2×10^{-6}
20×20	L_2	6.6×10^{-7}	3.3×10^{-7}	1.4×10^{-6}
	L_∞	9.4×10^{-7}	4.7×10^{-7}	1.9×10^{-6}
	L_1	1.4×10^{-7}	7.2×10^{-8}	3.1×10^{-7}
40×40	L_2	1.6×10^{-7}	8.1×10^{-8}	3.5×10^{-7}
	L_∞	2.4×10^{-7}	1.2×10^{-7}	4.9×10^{-7}

Table 4: L_1, L_2, and L_∞ values of the norm of the difference between the exact and computed solutions to the smooth problem for the density (ρ), pressure (p), and the magnitude of the velocity ($|(u,v)|$) at $t = 0.20$ for the meshes characterized in Table 3. These values are computed according to the procedures outlined in §2.

coefficients for the density, pressure, and velocity magnitude. Those results are compiled in Tables 5 through 7. These tables catalogue values of the convergence rates (q) and coefficients ($\mathcal{A}$), based on different error norms, for the coarse-to-fine ratios, σ, of the various characteristic mesh length scales based on the areas given in Table 3. The convergence results in the highlighted rows demonstrate that the calculations achieved the theoretical second-order spatial convergence in all norms for all length characterizations. These values are virtually identical for all norms, which is indicative of the fact that these different norms are equivalent when gauging a completely smooth (C^∞) function. The equivalence of these values for the different length scale measures reflects the fact that these different measures are virtually identical, as shown in Table 3. The near-identical behavior for the different fields provides evidence that the hydrodynamics algorithm is properly implemented for all fields. Overall, this is compelling verification evidence that this Lagrangian hydrodynamics algorithm is accurately coded for smooth problems in the software that ran these calculations.

One caveat to this problem is that it tests only the *linear* characteristic fields in the governing equations. There are other smooth problems that exercise the nonlinear fields of the governing equations [3]; however, we have not yet investigated these more challenging problems.

Smooth Problem Cell-Centered Density Results

100–400 Cells

	Area/N_c	Mean	Min	Max	Median
σ	2.00	2.00	2.00	2.00	2.00
L_1 rate	2.1	2.1	2.1	2.1	2.1
L_1 coeff.	2.8×10^{-4}	2.8×10^{-4}	2.8×10^{-4}	2.8×10^{-4}	2.8×10^{-4}
L_2 rate	2.1	2.1	2.1	2.1	2.1
L_2 coeff.	3.6×10^{-4}	3.6×10^{-4}	3.6×10^{-4}	3.6×10^{-4}	3.6×10^{-4}
L_∞ rate	2.0	2.0	2.0	2.0	2.0
L_∞ coeff.	3.9×10^{-4}	3.9×10^{-4}	3.9×10^{-4}	3.9×10^{-4}	3.9×10^{-4}

400–1600 Cells

	Area/N_c	Mean	Min	Max	Median
σ	2.00	2.00	2.00	2.00	2.00
L_1 rate	2.0	2.0	2.0	2.0	2.0
L_1 coeff.	2.5×10^{-4}	2.5×10^{-4}	2.5×10^{-4}	2.5×10^{-4}	2.5×10^{-4}
L_2 rate	2.0	2.0	2.0	2.0	2.0
L_2 coeff.	2.8×10^{-4}	2.8×10^{-4}	2.8×10^{-4}	2.8×10^{-4}	2.8×10^{-4}
L_∞ rate	2.0	2.0	2.0	2.0	2.0
L_∞ coeff.	3.5×10^{-4}	3.5×10^{-4}	3.5×10^{-4}	3.5×10^{-4}	3.5×10^{-4}

Table 5: L_1, L_2, and L_∞ values of the convergence rate (q) and convergence coefficient ($\mathcal{A}$) for the cell-centered density results for the smooth problem at $t = 0.20$. The top table catalogues the results computed on meshes with 100 and 400 cells, and the bottom table provides the results for meshes with 400 and 1600 cells. The results in the second through sixth columns were obtained with different methods by which to gauge the ratio of representative length scales that characterize the corresponding meshes. The values in columns two ("Area/N_c") through six ("Median") correspond to taking the ratio σ of the stated value for the coarse mesh divided by the corresponding value for the fine mesh. The values highlighted in gray boxes are approporiate convergence rates for this problem. The results for a given characteristic (e.g., L_1 rate) on a given pair of meshes (e.g., 100–400 cells) are identical for the ratios of different length measures to the two significant figures quoted; differences among certain values appeared in the third or higher significant figure.

Smooth Problem Cell-Centered Pressure Results

100–400 Cells

	Area/N_c	Mean	Min	Max	Median
σ	2.00	2.00	2.00	2.00	2.00
L_1 rate	2.1	2.1	2.1	2.1	2.1
L_1 coeff.	1.4×10^{-4}	1.4×10^{-4}	1.4×10^{-4}	1.4×10^{-4}	1.4×10^{-4}
L_2 rate	2.1	2.1	2.1	2.1	2.1
L_2 coeff.	1.8×10^{-4}	1.8×10^{-4}	1.8×10^{-4}	1.8×10^{-4}	1.8×10^{-4}
L_∞ rate	2.0	2.0	2.0	2.0	2.0
L_∞ coeff.	1.9×10^{-4}	1.9×10^{-4}	1.9×10^{-4}	1.9×10^{-4}	1.9×10^{-4}

400–1600 Cells

	Area/N_c	Mean	Min	Max	Median
σ	2.00	2.00	2.00	2.00	2.00
L_1 rate	2.0	2.0	2.0	2.0	2.0
L_1 coeff.	1.2×10^{-4}	1.2×10^{-4}	1.2×10^{-4}	1.2×10^{-4}	1.2×10^{-4}
L_2 rate	2.0	2.0	2.0	2.0	2.0
L_2 coeff.	1.4×10^{-4}	1.4×10^{-4}	1.4×10^{-4}	1.4×10^{-4}	1.4×10^{-4}
L_∞ rate	1.9	1.9	1.9	1.9	1.9
L_∞ coeff.	1.6×10^{-4}	1.6×10^{-4}	1.6×10^{-4}	1.6×10^{-4}	1.6×10^{-4}

Table 6: L_1, L_2, and L_∞ values of the convergence rate (q) and convergence coefficient ($\mathcal{A}$) for the cell-centered pressure results for the smooth problem at $t = 0.20$. The top table catalogues the results computed on meshes with 100 and 400 cells, and the bottom table provides the results for meshes with 400 and 1600 cells. The results in the second through sixth columns were obtained with different methods by which to gauge the ratio of representative length scales that characterize the corresponding meshes. The values in columns two ("Area/N_c") through six ("Median") correspond to taking the ratio σ of the stated value for the coarse mesh divided by the corresponding value for the fine mesh. The values highlighted in gray boxes are approporiate convergence rates for this problem. The results for a given characteristic (e.g., L_2 rate) on a given pair of meshes (e.g., 400–1600 cells) are identical for the ratios of different length measures to the two significant figures quoted; differences among certain values appeared in the third or higher significant figure.

Smooth Problem Vertex-Centered Velocity Results

121-441 Vertices

	Area/N_c	Mean	Min	Max	Median
σ	2.00	2.00	2.00	2.00	2.00
L_1 rate	2.0	2.0	2.0	2.0	2.0
L_1 coeff.	4.9×10^{-4}	4.9×10^{-4}	4.9×10^{-4}	4.9×10^{-4}	4.9×10^{-4}
L_2 rate	2.0	2.0	2.0	2.0	2.0
L_2 coeff.	4.9×10^{-4}	4.9×10^{-4}	4.9×10^{-4}	4.9×10^{-4}	4.9×10^{-4}
L_∞ rate	2.0	2.0	2.0	2.0	2.0
L_∞ coeff.	7.0×10^{-4}	7.0×10^{-4}	7.0×10^{-4}	7.0×10^{-4}	7.0×10^{-4}

441-1681 Vertices

	Area/N_c	Mean	Min	Max	Median
σ	2.00	2.00	2.00	2.00	2.00
L_1 rate	2.0	2.0	2.0	2.0	2.0
L_1 coeff.	4.7×10^{-4}	4.7×10^{-4}	4.7×10^{-4}	4.7×10^{-4}	4.7×10^{-4}
L_2 rate	2.0	2.0	2.0	2.0	2.0
L_2 coeff.	5.4×10^{-4}	5.4×10^{-4}	5.4×10^{-4}	5.4×10^{-4}	5.4×10^{-4}
L_∞ rate	2.0	2.0	2.0	2.0	2.0
L_∞ coeff.	7.6×10^{-4}	7.6×10^{-4}	7.6×10^{-4}	7.6×10^{-4}	7.6×10^{-4}

Table 7: L_1, L_2, and L_∞ values of the convergence rate (q) and convergence coefficient ($\mathcal{A}$) for the magnitude of the vertex-centered velocity for the smooth problem at $t = 0.20$. The top table catalogues the results computed on meshes with 121 and 441 vertices, and the bottom table provides the results for meshes with 441 and 1681 vertieces. The results in the second through sixth columns were obtained with different methods by which to gauge the ratio of representative length scales that characterize the corresponding meshes. The values in columns two ("Area/N_c") through six ("Median") correspond to taking the ratio σ of the stated value for the coarse mesh divided by the corresponding value for the fine mesh. The values highlighted in gray boxes are approporiate convergence rates for this problem. The results for a given characteristic (e.g., L_∞ rate) on a given pair of meshes (e.g., 121–441 vertices) are identical for the ratios of different length measures to the two significant figures quoted; differences among certain values appeared in the third or higher significant figure.

4 Convergence Analysis of a Non-Smooth Problem

In this section, we again employ the techniques utilized in the previous section to analyze 2-D hydrodynamics calculations on non-uniform meshes using a Lagrangian formation of the governing equations. We now consider a non-smooth problem, which we require to have an exactly-computable solution; for the sake of simplicity, we choose the spherically symmetric Noh problem [13, 14], computed on a mesh in 2-D cylindrical coordinates. The Noh problem, which admits a closed-form solution, is primarily a gauge of a hydrocode's ability to transform kinetic energy into internal energy. The global measure of convergence is again the asymptotic convergence rate. As for the smooth problem of the previous section, the hydrodynamics code being exercised here is part of the ASCI Shavano project, with the numerical approach being a control volume, staggered-grid, compatible, Lagrangian method [2, 4]. For this problem, a flux-limited artificial viscosity term is added to capture shock compressions.

4.1 Exact Solution of the Noh Problem

The Noh problem [13, 14] is a configuration for the Euler equations of gas dynamics in which the initial density is uniform, the initial specific internal energy is negligible, and the initial velocity is uniform and directed toward the origin, which is a reflecting boundary. The canonical configuration assumes a polytropic gas of adiabatic index $\gamma = 5/3$, with initial density and initial speed both unity; this problem has also been considered for materials with other equations of state [1]. With these initial conditions, given in Table 8, this configuration leads to a shock of infinite strength reflecting from the origin.

Assuming a semi-infinite domain, the Noh problem admits a closed-form, self-similar solution for a polytropic gas equation of state. There exists a closed-form solution for this problem having one-dimensional Cartesian, cylindrical, or spherical symmetry; we consider only the spherically symmetric case. For the polytropic gas equation of state, the solution at position r and time t is given by the following relations, in which d identifies the geometry of the problem (1 for Cartesian, 2 for cylindrical, and 3 for spherical), ρ_0 is the uniform initial density, and u_0 is the uniform initial velocity [7]:

$$\left\{\rho, e, u\right\} = \begin{cases} \left\{\rho_0 \left(\frac{\gamma+1}{\gamma-1}\right)^d, \frac{1}{2}u_0^2, 0\right\}, & \text{if } r < r_S \ , \\ \left\{\rho_0 \left[1 - (u_0 t/r)\right]^{d-1}, 0, u_0\right\}, & \text{if } r > r_S \ , \end{cases} \tag{40}$$

where the shock position r_S is given by

$$r_S = U_S\, t \qquad \text{with shock speed} \qquad U_S = \frac{1}{2}\,(\gamma - 1)\,|u_0|\,. \tag{41}$$

Spherical Noh Problem Initial Conditions

	ρ_0	p_0	e_0	u_0
$r > 0$	1	0	0	-1

Table 8: For each range of the spherical radial coordinate r, given are the initial values of the density ρ_0, pressure p_0, specific internal energy e_0, and radial velocity u_0 for the standard spherical Noh problem.

Spherical Noh Problem Solution at $t = 0.6$

	ρ	p	e	u
$r < r_S = 0.2$	64	$21\frac{1}{3}$	1/2	0
$r > r_S = 0.2$	$[1 + (0.6/r)]^2$	0	0	-1

Table 9: For each range of the spherical radial coordinate r, either less than or greater than the shock position r_S, given are the density ρ, pressure p, specific internal energy e, and radial velocity u for the standard spherical Noh problem.

The pressure is obtained from the above results together with the polytropic equation of state $p = (\gamma - 1)\, \rho\, e$. With the initial conditions prescribed above, the final time for the standard Noh problem used for comparison purposes is taken to be $t = 0.6$, following the Noh's original publication. Table 9 contains the closed-form solution for the spherically-symmetric Noh problem at this time.

4.2 Computed Solution of the Noh Problem

The numerical solution for the spherical Noh problem was obtained in the case of two-dimensional cylindrical geometry, with the axial coordinate z and cylindrical radial coordinate r $(\equiv \sqrt{x^2 + y^2}\,)$. Only the first quadrant of the (r, z)-plane was considered, with the initial grid tesselated by cells determined by K radial lines emanating from the origin (between and including $\theta = 0$ and $\theta = \pi/2$) and $L - 1$ circular arcs of nonzero radius centered at the origin, together with the point at the origin. The configurations considered consisted of $(K, L) = (16, 26)$, $(31, 51)$, and $(61, 101)$, corresponding to 375, 1500, and 6000 cells with 401, 1551, and 6101 vertices, respectively.

The initial conditions correspond to those listed in Table 8. The problems were run to a final time of 0.6, with uniform, constant timesteps. Previous experience has led us to believe that it is imperative that the final time of the calculations being compared be precise. Consequently, to ensure that the final simulation time was both accurate and consistent across calculations, the timestep for each of these computations was identical. This timestep was slightly

2-D Noh Problem Mesh Statistics

	Mesh 1	Mesh 2	Mesh 3
N_c	375	1500	6000
N_v	401	1551	6101
Area/N_c	3.34×10^{-4}	8.37×10^{-5}	2.09×10^{-5}
Mean	3.34×10^{-4}	8.37×10^{-5}	2.09×10^{-5}
Minimum	1.88×10^{-5}	2.21×10^{-6}	2.67×10^{-7}
Maximum	1.59×10^{-3}	4.08×10^{-4}	1.03×10^{-4}
Median	1.70×10^{-4}	3.70×10^{-5}	8.72×10^{-6}

Table 10: Statistics for the Noh problem meshes at the three resolutions considered: the number of cells, the number of vertices, the total area in the mesh divided by the number of cells, and the mean, minimum, maximum, and median of the areas of all cells in the mesh.

smaller than the minimum value calculated for the 61×101 mesh using the default CFL limit. Consequently, this constraint implied that the effective CFL number for the coarser calculations was well below that which would have otherwise been used for these calculations.

Table 10 contains information about the meshes at the final computational time. We provide values for various characteristics corresponding to the the total mesh area divided by the number of cells, and the minimum, maximum, mean, median over the set of all cell areas in the (r, z)-computational plane. Unlike the previous smooth problem, there is a significant difference between the minimum and maximum zone sizes, because of both the distortion of the mesh with the calculation and the convergence of the initial mesh at the origin. The variation in values in this table is suggestive of the challenge that Lagrangian codes present in the assignment of a truly representative length scale for the mesh cells.

4.3 Convergence Analysis of the Noh Problem

An ASCI Shavano project code was used to compute numerical solutions to the compressible gas dynamics equations for the problem described above. We consider asymptotic convergence analysis for two cell-centered quantities, the density and pressure, together with the vertex-centered velocity field. In this section, we describe the outcome of this study and catalogue the results in several tables.

The computed norms of the difference between calculated and exact solutions over the computational meshes (as characterized in Table 10) are provided in Table 11. The problem we consider here is formulated in 2-D cylindrically-

2-D Noh Problem Error Norms

		ρ	p	$\lvert(u,v)\rvert$
	L_1	6.0	2.1	1.3×10^{-1}
Mesh 1	L_2	1.2×10^{1}	4.7	3.1×10^{-1}
	L_∞	5.5×10^{1}	1.6×10^{1}	9.9×10^{-1}
	L_1	3.4	1.1	6.7×10^{-2}
Mesh 2	L_2	8.7	3.5	2.3×10^{-1}
	L_∞	5.4×10^{1}	1.8×10^{1}	1.0
	L_1	1.8	6.1×10^{-1}	3.1×10^{-2}
Mesh 3	L_2	6.2	2.6	1.5×10^{-1}
	L_∞	5.3×10^{1}	2.0×10^{1}	1.0

Table 11: L_1, L_2, and L_∞ values of the norm of the difference between the exact and computed solutions to the Noh problem for the density (ρ), pressure (p), and the magnitude of the velocity ($\lvert(u,v)\rvert$) at $t = 0.6$ for the meshes characterized in Table 10. These values are computed according to the procedures outlined in §2.

symmetric geometry, unlike the smooth problem discussed in §2.3. Consequently, the norm here is obtained as an estimate of a volume quadrature using the data in (r, z)-plane; that is, the volume element in the quadrature is taken to be $2\pi\, r\, dr\, dz$,[4] with the value of r in this expression taken to be the cell-centered r-coordinate.

The values of the L_1 and L_2 norms decrease as a function of increasing mesh resolution for these calculations, while the value of the L_∞ norm is nearly independent of mesh resolution. From these trends we immediately infer that the computed solution is indeed converging in the L_1 and L_2 norms and not converging in the L_∞ norm. The non-convergent behavior suggested by the L_∞ norm values is not unexpected for this problem containing a shockwave.

The inferences of convergent behavior are made precise in Tables 12, 13, and 14, which catalogue the convergence parameters for the computed values of density, pressure, and velocity magnitude on pairs of meshes when compared with the corresponding exact solution values. In these tables, the quantities in each column correspond to a chosen value of the ratio, σ, of coarse-to-fine characteristic length associated with that calculation, as shown in the second row. We highlight the convergence rates calculated using the square root of the area divided by the number of cells (second column), and the mean (third column) and median (sixth column) over the set of zone sizes for the entire computational mesh at the final time. We believe that these highlighted convergence

[4] The cylindrical-coordinate *volume* element, $2\pi\, r\, dr\, dz$, differs from the planar Cartesian *area* element, $dx\, dy$, used in the quadratures for the previously considered smooth problem.

rates are perhaps the best representatives of the computations.

The results in Tables 12–14 exhibit several features of interest. For example, the convergence rates vary with the choice of characteristic length scale. This result is a direct consequence of the variation in these values due to nonuniformities in the mesh, as suggested in Table 10.

The convergence rates also vary significantly with the norm used to measure the error; this result is consistent with the fact that the exact solution to this problem is neither continuous nor differentiable. Recall that the previous problem, for which the exact solution is arbitrarily smooth (i.e., C^∞), exhibited convergence rates (of approximately two) that were all effectively the same, regardless of the norm used to gauge the error, as the results in Tables 5–7 show. Importantly for the case at hand, the computed solutions are converging at approximately first order in the L_1 norm; this finding is consistent with both theoretical and numerical results for Eulerian codes on problems containing shocks [6, 8]. These results also demonstrate that L_1 norm is the appropriate norm with which to gauge convergence for problems with discontinuities.

These quantitative results, together with those in §3.3, constitute compelling evidence that the hydrodynamics algorithm is properly implemented in the ASCI Shavano project code that performed these calculations.

Noh Problem Cell-Centered Density Results

375–1500 Cells

	Area/N_c	Mean	Min	Max	Median
σ	2.00	2.00	2.92	1.97	2.14
L_1 rate	0.80	0.80	0.52	0.81	0.72
L_1 coeff.	1.5×10^2	1.5×10^2	1.0×10^2	8.3×10^1	1.4×10^2
L_2 rate	0.46	0.46	0.30	0.46	0.41
L_2 coeff.	7.4×10^1	7.4×10^1	6.0×10^1	5.4×10^1	7.2×10^1
L_∞ rate	2.5×10^{-2}	2.5×10^{-2}	1.6×10^{-2}	2.5×10^{-2}	2.2×10^{-2}
L_∞ coeff.	6.0×10^1	6.0×10^1	6.0×10^1	5.9×10^1	6.0×10^1

1500–6000 Cells

	Area/N_c	Mean	Min	Max	Median
σ	2.00	2.00	2.87	1.99	2.10
L_1 rate	0.90	0.90	0.59	0.91	0.84
L_1 coeff.	2.4×10^2	2.4×10^2	1.6×10^2	1.2×10^2	2.5×10^2
L_2 rate	0.50	0.50	0.33	0.50	0.47
L_2 coeff.	9.1×10^1	9.1×10^1	7.4×10^1	6.2×10^1	9.4×10^1
L_∞ rate	1.4×10^{-2}	1.4×10^{-2}	9.3×10^{-3}	1.4×10^{-2}	1.3×10^{-2}
L_∞ coeff.	5.7×10^1	5.7×10^1	5.7×10^1	5.7×10^1	5.7×10^1

Table 12: L_1, L_2, and L_∞ values of the convergence rate (q) and convergence coefficient ($\mathcal{A}$) for the cell-centered density results for the Noh problem at $t = 0.6$. The top table catalogues the results computed on meshes with 375 and 1500 cells, and the bottom table provides the results for meshes with 1500 and 6000 cells. The results in the second through sixth columns were obtained with different methods by which to gauge the ratio of representative length scales that characterize the corresponding meshes. The values in columns two ("Area/N_c") through six ("Median") correspond to taking the ratio σ of the stated value for the coarse mesh divided by the corresponding value for the fine mesh. The values highlighted in gray boxes are perhaps the most appropriate convergence rates for this problem.

Noh Problem Cell-Centered Pressure Results

375–1500 Cells

	Area/N_c	Mean	Min	Max	Median
σ	2.00	2.00	2.92	1.97	2.14
L_1 rate	0.86	0.86	0.56	0.88	0.78
L_1 coeff.	6.5×10^1	6.5×10^1	4.3×10^1	3.5×10^1	6.1×10^1
L_2 rate	0.42	0.42	0.27	0.42	0.38
L_2 coeff.	2.5×10^1	2.5×10^1	2.0×10^1	1.8×10^1	2.4×10^1
L_∞ rate	-2.2×10^{-1}	-2.2×10^{-1}	-1.4×10^{-1}	-2.2×10^{-1}	-2.0×10^{-1}
L_∞ coeff.	6.4	6.4	7.2	7.6	6.6

1500–6000 Cells

	Area/N_c	Mean	Min	Max	Median
σ	2.00	2.00	2.87	1.99	2.10
L_1 rate	0.92	0.92	0.60	0.93	0.86
L_1 coeff.	8.7×10^1	8.7×10^1	5.9×10^1	4.3×10^1	9.2×10^1
L_2 rate	0.46	0.46	0.30	0.46	0.43
L_2 coeff.	3.0×10^1	3.0×10^1	2.5×10^1	2.1×10^1	3.1×10^1
L_∞ rate	-1.4×10^{-1}	-1.4×10^{-1}	-8.9×10^{-2}	-1.4×10^{-1}	-1.3×10^{-1}
L_∞ coeff.	9.6	9.6	1.0×10^1	1.1×10^1	9.6

Table 13: L_1, L_2, and L_∞ values of the convergence rate (q) and convergence coefficient ($\mathcal{A}$) for the cell-centered pressure results for the Noh problem at $t = 0.6$. The top table catalogues the results computed on meshes with 375 and 1500 cells, and the bottom table provides the results for meshes with 1500 and 6000 cells. The results in the second through sixth columns were obtained with different methods by which to gauge the ratio of representative length scales that characterize the corresponding meshes. The values in columns two ("Area/N_c") through six ("Median") correspond to taking the ratio σ of the stated value for the coarse mesh divided by the corresponding value for the fine mesh. The values highlighted in gray boxes are perhaps the most appropriate convergence rates for this problem.

Noh Problem Vertex-Centered Velocity Results

401–1551 Vertices

	Area/N_c	Mean	Min	Max	Median
σ	2.00	2.00	2.92	1.97	2.14
L_1 rate	1.0	1.0	0.65	1.0	0.91
L_1 coeff.	7.6	7.6	4.7	3.7	7.1
L_2 rate	0.45	0.45	0.29	0.46	0.41
L_2 coeff.	1.9	1.9	1.5	1.4	1.8
L_∞ rate	-1.6×10^{-2}	-1.6×10^{-2}	-1.0×10^{-2}	-1.6×10^{-2}	-1.4×10^{-2}
L_∞ coeff.	0.93	0.93	0.94	0.94	0.93

1551–6101 Vertices

	Area/N_c	Mean	Min	Max	Median
σ	2.00	2.00	2.87	1.99	2.10
L_1 rate	1.1	1.1	0.72	1.1	1.0
L_1 coeff.	1.2×10^1	1.2×10^1	7.6	5.3	1.3×10^1
L_2 rate	0.54	0.54	0.36	0.55	0.51
L_2 coeff.	2.9	2.9	2.3	1.9	3.0
L_∞ rate	-9.6×10^{-3}	-9.6×10^{-3}	-6.3×10^{-3}	-9.7×10^{-3}	-9.0×10^{-3}
L_∞ coeff.	0.96	0.96	0.96	0.96	0.96

Table 14: L_1, L_2, and L_∞ values of the convergence rate (q) and convergence coefficient ($\mathcal{A}$) for the magnitude of the vertex-centered velocity for the Noh problem at $t = 0.6$. The top table catalogues the results computed on meshes with 401 and 1551 vertices, and the bottom table provides the results for meshes with 1551 and 6101 vertieces. The results in the second through sixth columns were obtained with different methods by which to gauge the ratio of representative length scales that characterize the corresponding meshes. The values in columns two ("Area/N_c") through six ("Median") correspond to taking the ratio σ of the stated value for the coarse mesh divided by the corresponding value for the fine mesh. The values highlighted in gray boxes are perhaps the most appropriate convergence rates for this problem.

5 Summary

This report describes a method for conducting asymptotic convergence analysis for a 2-D Lagrangian, compressible hydrodynamics algorithm based on an unstructured mesh. The documented results of such a study provide a foundational element of verification analysis for the numerical solution of discretized PDEs. Convergence analysis naturally suggests that the asymptotic convergence rate be highlighted as a principal gauge of code verification.

The two main complications of this analysis for a Lagrangian code are the irregular cell geometries and the dynamic evolution of the cells that constitute the mesh. In §2, we describe how to account for this increased complexity when estimating the asymptotic convergence parameters. In particular, we explain the procedures by which convergence rates and coefficients are obtained for both cell-centered and vertex-centered variables. There is no single unambiguous measure of the characteristic length corresponding to the cells in a non-uniform mesh; such meshes appear, e.g., in Lagrangian, AMR, and ALE calculations. We propose several possible such measures and employ them in the subsequent analysis.

In §3, we describe the results of this analysis for computed solutions of a 2-D, Cartesian geometry, smooth flow problem with periodic boundary conditions. The velocity and density/pressure perturbations for this problem are initially out of phase and remain so for the entire calculation, oscillating as a standing wave in the computational domain. By the nature of this smooth problem, the various measures with which to gauge the characteristic length scale of the computational mesh are virtually identical. The closed-form solution obtained is an exact solution of the full compressible flow equations only in the case of vanishingly small initial perturbation amplitude; numerically, we find this idealized behavior to be exhibited for non-dimensional velocity perturbation amplitudes of order 10^{-4}. Convergence analysis conducted on the computed results shows that the software implementation achieves the theoretical second-order spatial convergence for the cell-centered density and pressure as well as the vertex-centered velocity vector magnitude in all norms (i.e., L_1, L_2, and L_∞), consistent with the smooth nature of this problem.

The second case, in §4, consists of the well-known Noh problem, which describes the self-similar, spherically symmetric flow of a compressible, polytropic gas that is initially flowing uniformly inward toward a reflective origin. We consider this problem for the case of two-dimensional cylindrical (r, z) coordinates on three different meshes. The analyses of the cell-centered density and pressure as well as the vertex-centered velocity magnitude demonstrate approximately first-order convergence in the L_1 norm when the total mesh area divided by the number of cells as well as the mean and median of the mesh cell sizes are used to define a characteristic length. The L_2 and L_∞ results are of lower order, consistent with the notion that these norms are not the appropriate yardstick by which to measure numerical error for problems that contain discontinuous features (e.g., shocks). These results exhibit the behavior anticipated for a compressible hydrodynamics algorithms in the presence of discontinuities. Moreover, they are fully comparable to the results for Eulerian code calculations of problems with shocks and to the results for 1-D Lagrangian calculations of a Riemann shock-tube problem [8].

In this report, we demonstrate the viability of asymptotic converence analysis for a 2-D Lagrangian compressible hydrodynamics algorithm based on an unstructured mesh. Although we consider only two idealized problems (one with a smooth solution, one with a discontinuous solution), this approach could be extended to calculation verification of problems for which no exactly computable solution exists (see, e.g., [17]). Such analyses, which would require the development and application of software to accurately and conservatively interpolate data between disparate unstructured meshes (see, e.g., [12]), would provide a valuable contribution to the body of evidence desired for the verification of Lagrangian hydrocodes.

Acknowledgments

This work was performed by Los Alamos National Laboratory, which is operated by the University of California for the U.S. Department of Energy under contract W-7405-ENG-36. The helpful discussions with Ralph Nelson and Bill Chandler are gratefully acknowledged.

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